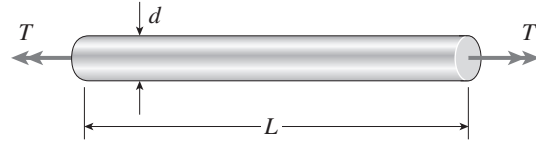


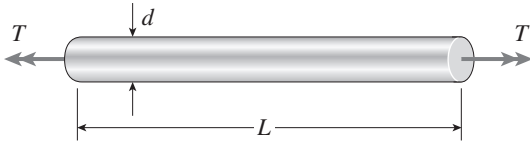
### Strain Energy in Torsion

**Problem 3.9-1** A solid circular bar of steel ( $G = 11.4 \times 10^6$  psi) with length  $L = 30$  in. and diameter  $d = 1.75$  in. is subjected to pure torsion by torques  $T$  acting at the ends (see figure).

- Calculate the amount of strain energy  $U$  stored in the bar when the maximum shear stress is 4500 psi.
- From the strain energy, calculate the angle of twist  $\phi$  (in degrees).



#### Solution 3.9-1 Steel bar



$$G = 11.4 \times 10^6 \text{ psi}$$

$$L = 30 \text{ in.}$$

$$d = 1.75 \text{ in.}$$

$$\tau_{\max} = 4500 \text{ psi}$$

$$\tau_{\max} = \frac{16 T}{\pi d^3} \quad T = \frac{\pi d^3 \tau_{\max}}{16} \quad (\text{Eq. 1})$$

$$I_P = \frac{\pi d^4}{32}$$

#### (a) STRAIN ENERGY

$$\begin{aligned} U &= \frac{T^2 L}{2GI_P} = \left( \frac{\pi d^3 \tau_{\max}}{16} \right)^2 \left( \frac{L}{2G} \right) \left( \frac{32}{\pi d^4} \right) \\ &= \frac{\pi d^2 L \tau_{\max}^2}{16G} \end{aligned} \quad (\text{Eq. 2})$$

Substitute numerical values:

$$U = 32.0 \text{ in.-lb} \quad \leftarrow$$

#### (b) ANGLE OF TWIST

$$U = \frac{T\phi}{2} \quad \phi = \frac{2U}{T}$$

Substitute for  $T$  and  $U$  from Eqs. (1) and (2):

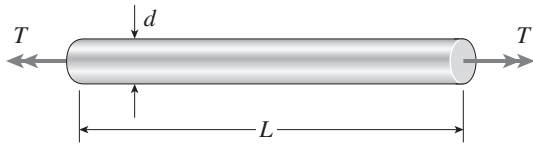
$$\phi = \frac{2L\tau_{\max}}{Gd} \quad (\text{Eq. 3})$$

Substitute numerical values:

$$\phi = 0.013534 \text{ rad} = 0.775^\circ \quad \leftarrow$$

**Problem 3.9-2** A solid circular bar of copper ( $G = 45$  GPa) with length  $L = 0.75$  m and diameter  $d = 40$  mm is subjected to pure torsion by torques  $T$  acting at the ends (see figure).

- Calculate the amount of strain energy  $U$  stored in the bar when the maximum shear stress is 32 MPa.
- From the strain energy, calculate the angle of twist  $\phi$  (in degrees)

**Solution 3.9-2** Copper bar

$$G = 45 \text{ GPa}$$

$$L = 0.75 \text{ m}$$

$$d = 40 \text{ mm}$$

$$\tau_{\max} = 32 \text{ MPa}$$

$$\tau_{\max} = \frac{16T}{\pi d^3} \quad T = \frac{\pi d^3 \tau_{\max}}{16} \quad (\text{Eq. 1})$$

$$I_P = \frac{\pi d^4}{32}$$

## (a) STRAIN ENERGY

$$U = \frac{T^2 L}{2GI_P} = \left( \frac{\pi d^3 \tau_{\max}}{16} \right)^2 \left( \frac{L}{2G} \right) \left( \frac{32}{\pi d^4} \right)$$

$$= \frac{\pi d^2 L \tau_{\max}^2}{16G} \quad (\text{Eq. 2})$$

Substitute numerical values:

$$U = 5.36 \text{ J} \quad \leftarrow$$

## (b) ANGLE OF TWIST

$$U = \frac{T\phi}{2} \quad \phi = \frac{2U}{T}$$

Substitute for  $T$  and  $U$  from Eqs. (1) and (2):

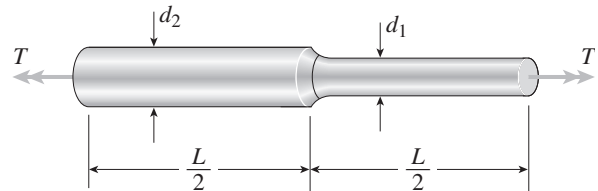
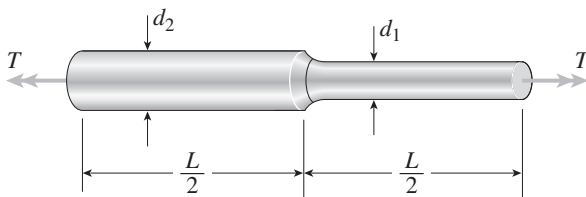
$$\phi = \frac{2L\tau_{\max}}{Gd} \quad (\text{Eq. 3})$$

Substitute numerical values:

$$\phi = 0.026667 \text{ rad} = 1.53^\circ \quad \leftarrow$$

**Problem 3.9-3** A stepped shaft of solid circular cross sections (see figure) has length  $L = 45$  in., diameter  $d_2 = 1.2$  in., and diameter  $d_1 = 1.0$  in. The material is brass with  $G = 5.6 \times 10^6$  psi.

Determine the strain energy  $U$  of the shaft if the angle of twist is  $3.0^\circ$ .

**Solution 3.9-3** Stepped shaft

$$d_1 = 1.0 \text{ in.}$$

$$d_2 = 1.2 \text{ in.}$$

$$L = 45 \text{ in.}$$

$$G = 5.6 \times 10^6 \text{ psi (brass)}$$

$$\phi = 3.0^\circ = 0.0523599 \text{ rad}$$

## STRAIN ENERGY

$$U = \sum \frac{T^2 L}{2GI_P} = \frac{16 T^2 (L/2)}{\pi G d_2^4} + \frac{16 T^2 (L/2)}{\pi G d_1^4}$$

$$= \frac{8T^2 L}{\pi G} \left( \frac{1}{d_2^4} + \frac{1}{d_1^4} \right) \quad (\text{Eq. 1})$$

$$\text{Also, } U = \frac{T\phi}{2} \quad (\text{Eq. 2})$$

Equate  $U$  from Eqs. (1) and (2) and solve for  $T$ :

$$T = \frac{\pi G d_1^4 d_2^4 \phi}{16L(d_1^4 + d_2^4)}$$

$$U = \frac{T\phi}{2} = \frac{\pi G \phi^2}{32L} \left( \frac{d_1^4 d_2^4}{d_1^4 + d_2^4} \right) \quad \phi = \text{radians}$$

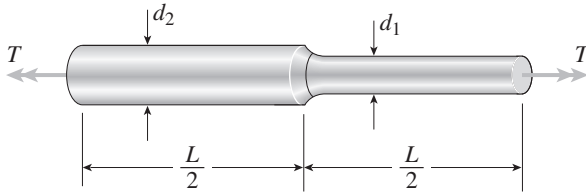
SUBSTITUTE NUMERICAL VALUES:

$$U = 22.6 \text{ in.-lb} \quad \leftarrow$$

**Problem 3.9-4** A stepped shaft of solid circular cross sections (see figure) has length  $L = 0.80$  m, diameter  $d_2 = 40$  mm, and diameter  $d_1 = 30$  mm. The material is steel with  $G = 80$  GPa.

Determine the strain energy  $U$  of the shaft if the angle of twist is  $1.0^\circ$ .

**Soluton 3.9-4 Stepped shaft**



$$\begin{aligned} d_1 &= 30 \text{ mm} & d_2 &= 40 \text{ mm} \\ L &= 0.80 \text{ m} & G &= 80 \text{ GPa (steel)} \\ \phi &= 1.0^\circ = 0.0174533 \text{ rad} \end{aligned}$$

STRAIN ENERGY

$$\begin{aligned} U &= \sum \frac{T^2 L}{2GI_p} = \frac{16T^2(L/2)}{\pi G d_2^4} + \frac{16T^2(L/2)}{\pi G d_1^4} \\ &= \frac{8T^2 L}{\pi G} \left( \frac{1}{d_2^4} + \frac{1}{d_1^4} \right) \end{aligned} \quad (\text{Eq. 1})$$

$$\text{Also, } U = \frac{T\phi}{2} \quad (\text{Eq. 2})$$

Equate  $U$  from Eqs. (1) and (2) and solve for  $T$ :

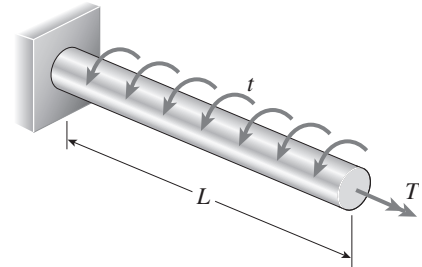
$$\begin{aligned} T &= \frac{\pi G d_1^4 d_2^4 \phi}{16L(d_1^4 + d_2^4)} \\ U &= \frac{T\phi}{2} = \frac{\pi G \phi^2}{32L} \left( \frac{d_1^4 d_2^4}{d_1^4 + d_2^4} \right) \quad \phi = \text{radians} \end{aligned}$$

SUBSTITUTE NUMERICAL VALUES:

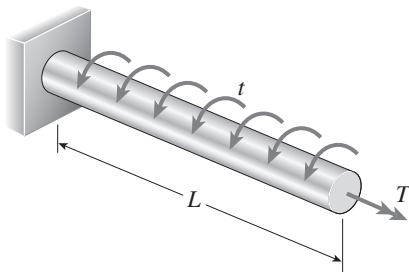
$$U = 1.84 \text{ J} \quad \leftarrow$$

**Problem 3.9-5** A cantilever bar of circular cross section and length  $L$  is fixed at one end and free at the other (see figure). The bar is loaded by a torque  $T$  at the free end and by a distributed torque of constant intensity  $t$  per unit distance along the length of the bar.

- What is the strain energy  $U_1$  of the bar when the load  $T$  acts alone?
- What is the strain energy  $U_2$  when the load  $t$  acts alone?
- What is the strain energy  $U_3$  when both loads act simultaneously?



**Solution 3.9-5 Cantilever bar with distributed torque**



$G$  = shear modulus

$I_p$  = polar moment of inertia

$T$  = torque acting at free end

$t$  = torque per unit distance

(a) LOAD  $T$  ACTS ALONE (Eq. 3-51a)

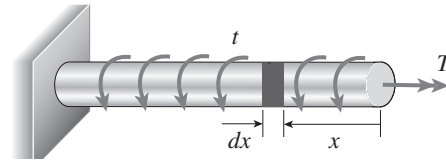
$$U_1 = \frac{T^2 L}{2GI_p} \quad \leftarrow$$

(b) LOAD  $t$  ACTS ALONE

From Eq. (3-56) of Example 3-11:

$$U_2 = \frac{t^2 L^3}{6GI_p} \quad \leftarrow$$

(c) BOTH LOADS ACT SIMULTANEOUSLY

At distance  $x$  from the free end:

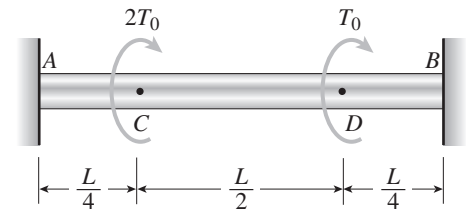
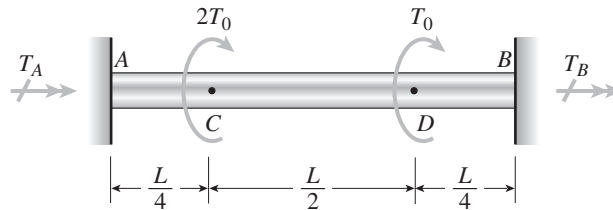
$$T(x) = T + tx$$

$$\begin{aligned} U_3 &= \int_0^L \frac{[T(x)]^2}{2GI_p} dx = \frac{1}{2GI_p} \int_0^L (T + tx)^2 dx \\ &= \frac{T^2 L}{2GI_p} + \frac{TtL^2}{2GI_p} + \frac{t^2 L^3}{6GI_p} \quad \leftarrow \end{aligned}$$

NOTE:  $U_3$  is *not* the sum of  $U_1$  and  $U_2$ .

**Problem 3.9-6** Obtain a formula for the strain energy  $U$  of the statically indeterminate circular bar shown in the figure. The bar has fixed supports at ends  $A$  and  $B$  and is loaded by torques  $2T_0$  and  $T_0$  at points  $C$  and  $D$ , respectively.

*Hint:* Use Eqs. 3-46a and b of Example 3-9, Section 3.8, to obtain the reactive torques.

**Solution 3.9-6** Statically indeterminate bar

## REACTIVE TORQUES

From Eq. (3-46a):

$$T_A = \frac{(2T_0)\left(\frac{3L}{4}\right)}{L} + \frac{T_0\left(\frac{L}{4}\right)}{L} = \frac{7T_0}{4}$$

$$T_B = 3T_0 - T_A = \frac{5T_0}{4}$$

## INTERNAL TORQUES

$$T_{AC} = -\frac{7T_0}{4} \quad T_{CD} = \frac{T_0}{4} \quad T_{DB} = \frac{5T_0}{4}$$

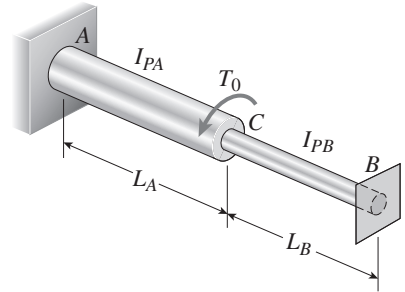
## STRAIN ENERGY (from Eq. 3-53)

$$\begin{aligned} U &= \sum_{i=1}^n \frac{T_i^2 L_i}{2G_i I_{pi}} \\ &= \frac{1}{2GI_p} \left[ T_{AC}^2 \left(\frac{L}{4}\right) + T_{CD}^2 \left(\frac{L}{2}\right) + T_{DB}^2 \left(\frac{L}{4}\right) \right] \\ &= \frac{1}{2GI_p} \left[ \left(-\frac{7T_0}{4}\right)^2 \left(\frac{L}{4}\right) + \left(\frac{T_0}{4}\right)^2 \left(\frac{L}{2}\right) + \left(\frac{5T_0}{4}\right)^2 \left(\frac{L}{4}\right) \right] \\ U &= \frac{19T_0^2 L}{32GI_p} \quad \leftarrow \end{aligned}$$

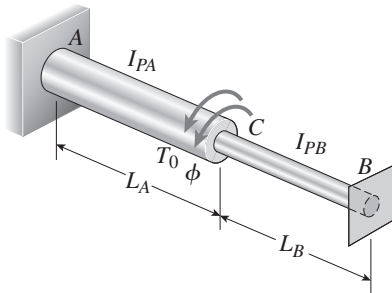
**Problem 3.9-7** A statically indeterminate stepped shaft  $ACB$  is fixed at ends  $A$  and  $B$  and loaded by a torque  $T_0$  at point  $C$  (see figure). The two segments of the bar are made of the same material, have lengths  $L_A$  and  $L_B$ , and have polar moments of inertia  $I_{PA}$  and  $I_{PB}$ .

Determine the angle of rotation  $\phi$  of the cross section at  $C$  by using strain energy.

*Hint:* Use Eq. 3-51b to determine the strain energy  $U$  in terms of the angle  $\phi$ . Then equate the strain energy to the work done by the torque  $T_0$ . Compare your result with Eq. 3-48 of Example 3-9, Section 3.8.



**Solution 3.9-7 Statically indeterminate bar**



STRAIN ENERGY (FROM EQ. 3-51B)

$$U = \sum_{i=1}^n \frac{GI_{Pi}\phi_i^2}{2L_i} = \frac{GI_{PA}\phi^2}{2L_A} + \frac{GI_{PB}\phi^2}{2L_B}$$

$$= \frac{G\phi^2}{2} \left( \frac{I_{PA}}{L_A} + \frac{I_{PB}}{L_B} \right)$$

WORK DONE BY THE TORQUE  $T_0$

$$W = \frac{T_0\phi}{2}$$

EQUATE  $U$  AND  $W$  AND SOLVE FOR  $\phi$

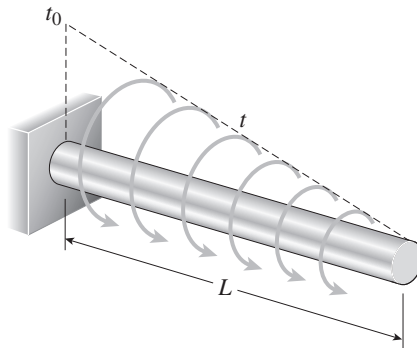
$$\frac{G\phi^2}{2} \left( \frac{I_{PA}}{L_A} + \frac{I_{PB}}{L_B} \right) = \frac{T_0\phi}{2}$$

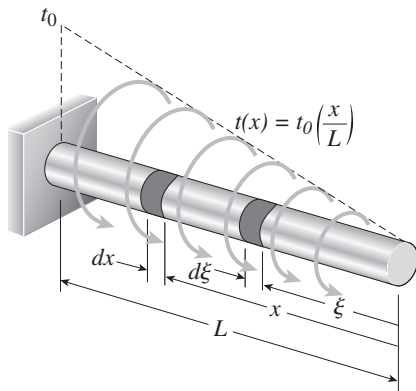
$$\phi = \frac{T_0L_AL_B}{G(L_BI_{PA} + L_AI_{PB})} \leftarrow$$

(This result agrees with Eq. (3-48) of Example 3-9, Section 3.8.)

**Problem 3.9-8** Derive a formula for the strain energy  $U$  of the cantilever bar shown in the figure.

The bar has circular cross sections and length  $L$ . It is subjected to a distributed torque of intensity  $t$  per unit distance. The intensity varies linearly from  $t = 0$  at the free end to a maximum value  $t = t_0$  at the support.

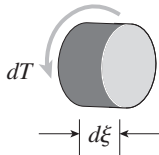


**Solution 3.9-8 Cantilever bar with distributed torque**

$x$  = distance from right-hand end of the bar

**ELEMENT  $d\xi$** 

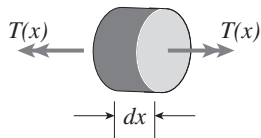
Consider a differential element  $d\xi$  at distance  $\xi$  from the right-hand end.



$dT$  = external torque acting on this element

$$dT = t(\xi)d\xi$$

$$= t_0\left(\frac{\xi}{L}\right)d\xi$$

**ELEMENT  $dx$  AT DISTANCE  $x$** 

$T(x)$  = internal torque acting on this element

$T(x)$  = total torque from  $x = 0$  to  $x = x$

$$\begin{aligned} T(x) &= \int_0^x dT = \int_0^x t_0\left(\frac{\xi}{L}\right)d\xi \\ &= \frac{t_0x^2}{2L} \end{aligned}$$

**STRAIN ENERGY OF ELEMENT  $dx$** 

$$\begin{aligned} dU &= \frac{[T(x)]^2 dx}{2GI_p} = \frac{1}{2GI_p} \left(\frac{t_0}{2L}\right)^2 x^4 dx \\ &= \frac{t_0^2}{8L^2GI_p} x^4 dx \end{aligned}$$

**STRAIN ENERGY OF ENTIRE BAR**

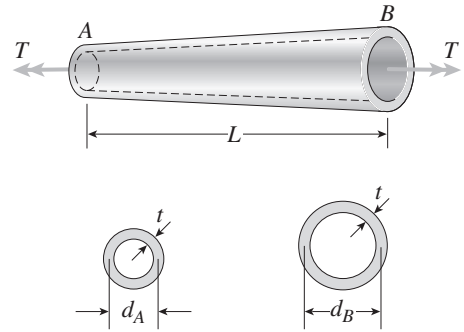
$$\begin{aligned} U &= \int_0^L dU = \frac{t_0^2}{8L^2GI_p} \int_0^L x^4 dx \\ &= \frac{t_0^2}{8L^2GI_p} \left(\frac{L^5}{5}\right) \end{aligned}$$

$$U = \frac{t_0^2 L^3}{40GI_p} \quad \leftarrow$$

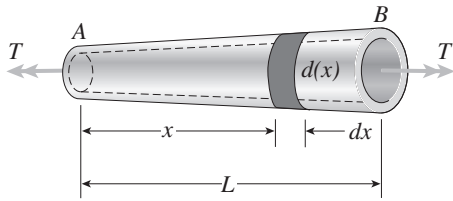
**Problem 3.9-9** A thin-walled hollow tube  $AB$  of conical shape has constant thickness  $t$  and average diameters  $d_A$  and  $d_B$  at the ends (see figure).

- (a) Determine the strain energy  $U$  of the tube when it is subjected to pure torsion by torques  $T$ .
- (b) Determine the angle of twist  $\phi$  of the tube.

*Note:* Use the approximate formula  $I_p \approx \pi d^3 t / 4$  for a thin circular ring; see Case 22 of Appendix D.



**Solution 3.9-9 Thin-walled, hollow tube**



- $t$  = thickness
- $d_A$  = average diameter at end A
- $d_B$  = average diameter at end B
- $d(x)$  = average diameter at distance  $x$  from end A
- $d(x) = d_A + \left(\frac{d_B - d_A}{L}\right)x$

POLAR MOMENT OF INERTIA

$$I_p = \frac{\pi d^3 t}{4}$$

$$I_p(x) = \frac{\pi [d(x)]^3 t}{4} = \frac{\pi t}{4} \left[ d_A + \left(\frac{d_B - d_A}{L}\right)x \right]^3$$

(a) STRAIN ENERGY (FROM EQ. 3-54)

$$U = \int_0^L \frac{T^2 dx}{2GI_p(x)}$$

$$= \frac{2T^2}{\pi Gt} \int_0^L \frac{dx}{\left[ d_A + \left(\frac{d_B - d_A}{L}\right)x \right]^3} \quad \text{(Eq. 1)}$$

From Appendix C:

$$\int \frac{dx}{(a + bx)^3} = -\frac{1}{2b(a + bx)^2}$$

Therefore,

$$\int_0^L \frac{dx}{\left[ d_A + \left(\frac{d_B - d_A}{L}\right)x \right]^3}$$

$$= -\frac{1}{\frac{2(d_B - d_A)}{L} \left[ d_A + \left(\frac{d_B - d_A}{L}\right)x \right]^2} \Bigg|_0^L$$

$$= -\frac{L}{2(d_B - d_A)(d_B)^2} + \frac{L}{2(d_B - d_A)(d_A)^2}$$

$$= \frac{L(d_A + d_B)}{2d_A^2 d_B^2}$$

Substitute this expression for the integral into the equation for  $U$  (Eq. 1):

$$U = \frac{2T^2}{\pi Gt} \cdot \frac{L(d_A + d_B)}{2d_A^2 d_B^2} = \frac{T^2 L}{\pi Gt} \left( \frac{d_A + d_B}{d_A^2 d_B^2} \right) \leftarrow$$

(b) ANGLE OF TWIST

Work of the torque  $T$ :  $W = \frac{T\phi}{2}$

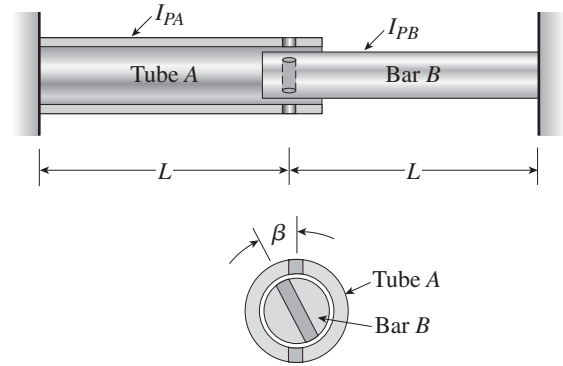
$$W = U \quad \frac{T\phi}{2} = \frac{T^2 L (d_A + d_B)}{\pi Gt d_A^2 d_B^2}$$

Solve for  $\phi$ :

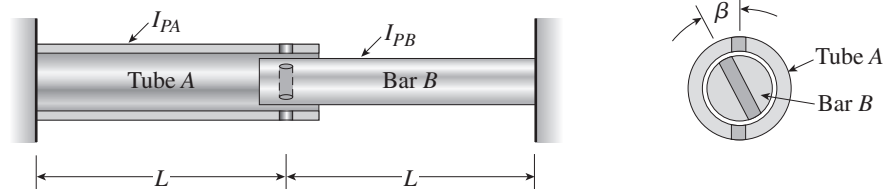
$$\phi = \frac{2TL(d_A + d_B)}{\pi Gt d_A^2 d_B^2} \leftarrow$$

**Problem 3.9-10** A hollow circular tube  $A$  fits over the end of a solid circular bar  $B$ , as shown in the figure. The far ends of both bars are fixed. Initially, a hole through bar  $B$  makes an angle  $\beta$  with a line through two holes in tube  $A$ . Then bar  $B$  is twisted until the holes are aligned, and a pin is placed through the holes.

When bar  $B$  is released and the system returns to equilibrium, what is the total strain energy  $U$  of the two bars? (Let  $I_{PA}$  and  $I_{PB}$  represent the polar moments of inertia of bars  $A$  and  $B$ , respectively. The length  $L$  and shear modulus of elasticity  $G$  are the same for both bars.)



### Solution 3.9-10 Circular tube and bar



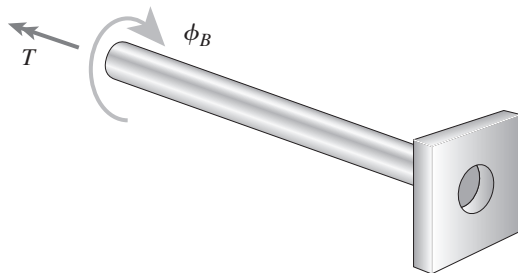
#### TUBE A



$T$  = torque acting on the tube

$\phi_A$  = angle of twist

#### BAR B



$T$  = torque acting on the bar

$\phi_B$  = angle of twist

#### COMPATIBILITY

$$\phi_A + \phi_B = \beta$$

#### FORCE-DISPLACEMENT RELATIONS

$$\phi_A = \frac{TL}{GI_{PA}} \quad \phi_B = \frac{TL}{GI_{PB}}$$

Substitute into the equation of compatibility and solve for  $T$ :

$$T = \frac{\beta G}{L} \left( \frac{I_{PA} I_{PB}}{I_{PA} + I_{PB}} \right)$$

#### STRAIN ENERGY

$$\begin{aligned} U &= \sum \frac{T^2 L}{2GI_P} = \frac{T^2 L}{2GI_{PA}} + \frac{T^2 L}{2GI_{PB}} \\ &= \frac{T^2 L}{2G} \left( \frac{1}{I_{PA}} + \frac{1}{I_{PB}} \right) \end{aligned}$$

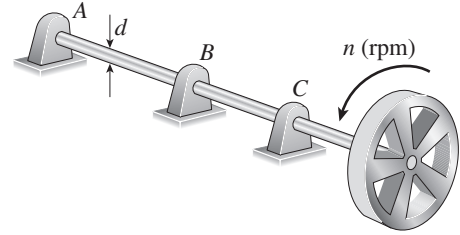
Substitute for  $T$  and simplify:

$$U = \frac{\beta^2 G}{2L} \left( \frac{I_{PA} I_{PB}}{I_{PA} + I_{PB}} \right) \quad \leftarrow$$

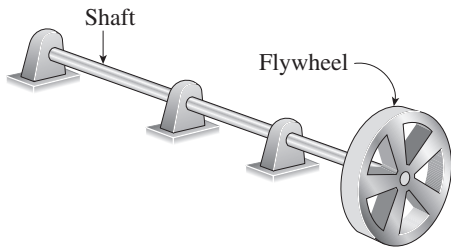
**Problem 3.9-11** A heavy flywheel rotating at  $n$  revolutions per minute is rigidly attached to the end of a shaft of diameter  $d$  (see figure). If the bearing at  $A$  suddenly freezes, what will be the maximum angle of twist  $\phi$  of the shaft? What is the corresponding maximum shear stress in the shaft?

(Let  $L$  = length of the shaft,  $G$  = shear modulus of elasticity, and  $I_m$  = mass moment of inertia of the flywheel about the axis of the shaft. Also, disregard friction in the bearings at  $B$  and  $C$  and disregard the mass of the shaft.)

*Hint:* Equate the kinetic energy of the rotating flywheel to the strain energy of the shaft.



### Solution 3.9-11 Rotating flywheel



$d$  = diameter

$n$  = rpm

KINETIC ENERGY OF FLYWHEEL

$$\text{K.E.} = \frac{1}{2} I_m \omega^2$$

$$\omega = \frac{2\pi n}{60}$$

$n$  = rpm

$$\begin{aligned} \text{K.E.} &= \frac{1}{2} I_m \left( \frac{2\pi n}{60} \right)^2 \\ &= \frac{\pi^2 n^2 I_m}{1800} \end{aligned}$$

UNITS:

$$I_m = (\text{force})(\text{length})(\text{second})^2$$

$\omega$  = radians per second

K.E. = (length)(force)

STRAIN ENERGY OF SHAFT (FROM EQ. 3-51b)

$$U = \frac{GI_p \phi^2}{2L}$$

$$I_p = \frac{\pi}{32} d^4$$

$d$  = diameter of shaft

$$U = \frac{\pi G d^4 \phi^2}{64L}$$

UNITS:

$G$  = (force)/(length)<sup>2</sup>

$I_p$  = (length)<sup>4</sup>

$\phi$  = radians

$L$  = length

$U$  = (length)(force)

EQUATE KINETIC ENERGY AND STRAIN ENERGY

$$\text{K.E.} = U \quad \frac{\pi^2 n^2 I_m}{1800} = \frac{\pi G d^4 \phi^2}{64L}$$

Solve for  $\phi$ :

$$\phi = \frac{2n}{15d^2} \sqrt{\frac{2\pi I_m L}{G}} \quad \leftarrow$$

MAXIMUM SHEAR STRESS

$$\tau = \frac{T(d/2)}{I_p} \quad \phi = \frac{TL}{GI_p}$$

Eliminate  $T$ :

$$\tau = \frac{Gd\phi}{2L}$$

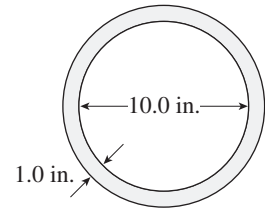
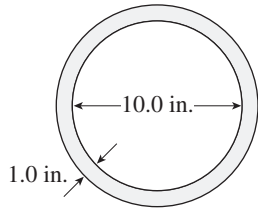
$$\tau_{\max} = \frac{Gd}{2L} \cdot \frac{2n}{15d^2} \sqrt{\frac{2\pi I_m L}{G}}$$

$$\tau_{\max} = \frac{n}{15d} \sqrt{\frac{2\pi G I_m}{L}} \quad \leftarrow$$

## Thin-Walled Tubes

**Problem 3.10-1** A hollow circular tube having an inside diameter of 10.0 in. and a wall thickness of 1.0 in. (see figure) is subjected to a torque  $T = 1200$  k-in.

Determine the maximum shear stress in the tube using (a) the approximate theory of thin-walled tubes, and (b) the exact torsion theory. Does the approximate theory give conservative or nonconservative results?

**Solution 3.10-1** Hollow circular tube

$$T = 1200 \text{ k-in.}$$

$$t = 1.0 \text{ in.}$$

$$r = \text{radius to median line}$$

$$r = 5.5 \text{ in.}$$

$$d_2 = \text{outside diameter} = 12.0 \text{ in.}$$

$$d_1 = \text{inside diameter} = 10.0 \text{ in.}$$

APPROXIMATE THEORY (EQ. 3-63)

$$\tau_1 = \frac{T}{2\pi r^2 t} = \frac{1200 \text{ k-in.}}{2\pi(5.5 \text{ in.})^2(1.0 \text{ in.})} = 6314 \text{ psi}$$

$$\tau_{\text{approx}} = 6310 \text{ psi} \quad \leftarrow$$

EXACT THEORY (EQ. 3-11)

$$\tau_2 = \frac{T(d_2/2)}{I_p} = \frac{Td_2}{2\left(\frac{\pi}{32}\right)d_2^4 - d_1^4}$$

$$= \frac{16(1200 \text{ k-in.})(12.0 \text{ in.})}{\pi[(12.0 \text{ in.})^4 - (10.0 \text{ in.})^4]}$$

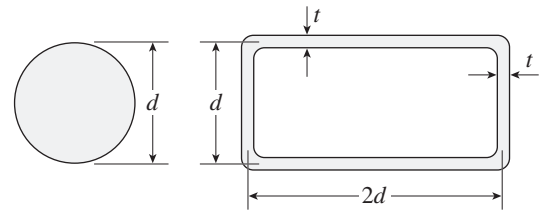
$$= 6831 \text{ psi}$$

$$\tau_{\text{exact}} = 6830 \text{ psi} \quad \leftarrow$$

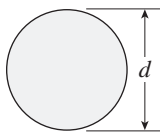
Because the approximate theory gives stresses that are too low, it is nonconservative. Therefore, the approximate theory should only be used for very thin tubes.

**Problem 3.10-2** A solid circular bar having diameter  $d$  is to be replaced by a rectangular tube having cross-sectional dimensions  $d \times 2d$  to the median line of the cross section (see figure).

Determine the required thickness  $t_{\text{min}}$  of the tube so that the maximum shear stress in the tube will not exceed the maximum shear stress in the solid bar.

**Solution 3.10-2** Bar and tube

SOLID BAR



$$\tau_{\text{max}} = \frac{16T}{\pi d^3} \quad (\text{Eq. 3-12})$$

$$A_m = (d)(2d) = 2d^2 \quad (\text{Eq. 3-64})$$

$$\tau_{\text{max}} = \frac{T}{2tA_m} = \frac{T}{4td^2} \quad (\text{Eq. 3-61})$$

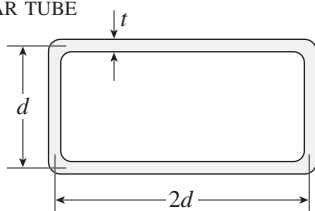
EQUATE THE MAXIMUM SHEAR STRESSES AND SOLVE FOR  $t$

$$\frac{16T}{\pi d^3} = \frac{T}{4td^2}$$

$$t_{\text{min}} = \frac{\pi d}{64} \quad \leftarrow$$

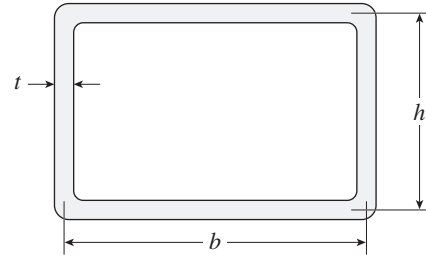
If  $t > t_{\text{min}}$ , the shear stress in the tube is less than the shear stress in the bar.

RECTANGULAR TUBE



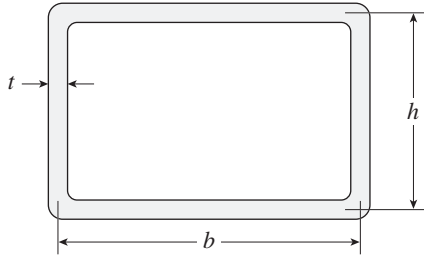
**Problem 3.10-3** A thin-walled aluminum tube of rectangular cross section (see figure) has a centerline dimensions  $b = 6.0$  in. and  $h = 4.0$  in. The wall thickness  $t$  is constant and equal to 0.25 in.

- (a) Determine the shear stress in the tube due to a torque  $T = 15$  k-in.
- (b) Determine the angle of twist (in degrees) if the length  $L$  of the tube is 50 in. and the shear modulus  $G$  is  $4.0 \times 10^6$  psi.



Use with Prob. 3.10-4

**Solution 3.10-3 Thin-walled tube**



- $b = 6.0$  in.
- $h = 4.0$  in.
- $t = 0.25$  in.
- $T = 15$  k-in.
- $L = 50$  in.
- $G = 4.0 \times 10^6$  psi

Eq. (3-64):  $A_m = bh = 24.0 \text{ in.}^2$

Eq. (3-71) with  $t_1 = t_2 = t$ :  $J = \frac{2b^2h^2t}{b+h}$

$J = 28.8 \text{ in.}^4$

(a) SHEAR STRESS (EQ. 3-61)

$\tau = \frac{T}{2tA_m} = 1250 \text{ psi} \leftarrow$

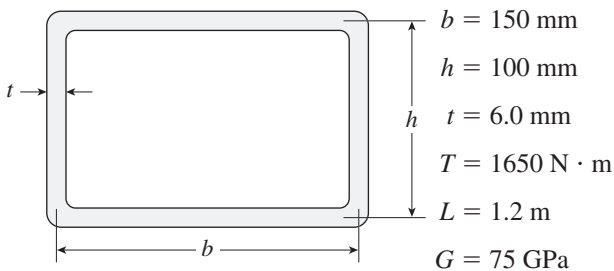
(b) ANGLE OF TWIST (EQ. 3-72)

$\phi = \frac{TL}{GJ} = 0.0065104 \text{ rad}$   
 $= 0.373^\circ \leftarrow$

**Problem 3.10-4** A thin-walled steel tube of rectangular cross section (see figure) has centerline dimensions  $b = 150$  mm and  $h = 100$  mm. The wall thickness  $t$  is constant and equal to 6.0 mm.

- (a) Determine the shear stress in the tube due to a torque  $T = 1650 \text{ N} \cdot \text{m}$ .
- (b) Determine the angle of twist (in degrees) if the length  $L$  of the tube is 1.2 m and the shear modulus  $G$  is 75 GPa.

**Solution 3.10-4 Thin-walled tube**



Eq. (3-64):  $A_m = bh = 0.015 \text{ m}^2$

Eq. (3-71) with  $t_1 = t_2 = t$ :  $J = \frac{2b^2h^2t}{b+h}$

$J = 10.8 \times 10^{-6} \text{ m}^4$

(a) SHEAR STRESS (Eq. 3-61)

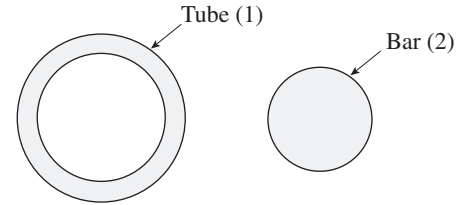
$\tau = \frac{T}{2tA_m} = 9.17 \text{ MPa} \leftarrow$

(b) ANGLE OF TWIST (Eq. 3-72)

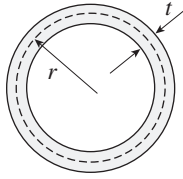
$\phi = \frac{TL}{GJ} = 0.002444 \text{ rad}$   
 $= 0.140^\circ \leftarrow$

**Problem 3.10-5** A thin-walled circular tube and a solid circular bar of the same material (see figure) are subjected to torsion. The tube and bar have the same cross-sectional area and the same length.

What is the ratio of the strain energy  $U_1$  in the tube to the strain energy  $U_2$  in the solid bar if the maximum shear stresses are the same in both cases? (For the tube, use the approximate theory for thin-walled bars.)



**Solution 3.10-5** THIN-WALLED TUBE (1)



$$A_m = \pi r^2 \quad J = 2\pi r^3 t \quad A = 2\pi r t$$

$$\tau_{\max} = \frac{T}{2tA_m} = \frac{T}{2\pi r^2 t}$$

$$T = 2\pi r^2 t \tau_{\max}$$

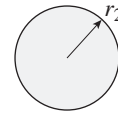
$$U_1 = \frac{T^2 L}{2GJ} = \frac{(2\pi r^2 t \tau_{\max})^2 L}{2G(2\pi r^3 t)}$$

$$= \frac{\pi r t \tau_{\max}^2 L}{G}$$

$$\text{But } r t = \frac{A}{2\pi}$$

$$\therefore U_1 = \frac{A \tau_{\max}^2 L}{2G}$$

SOLID BAR (2)



$$A = \pi r_2^2 \quad I_p = \frac{\pi}{2} r_2^4$$

$$\tau_{\max} = \frac{T r_2}{I_p} = \frac{2T}{\pi r_2^3} \quad T = \frac{\pi r_2^3 \tau_{\max}}{2}$$

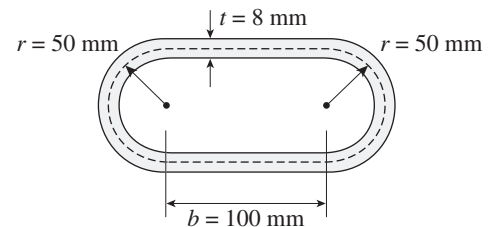
$$U_2 = \frac{T^2 L}{2GI_p} = \frac{(\pi r_2^3 \tau_{\max})^2 L}{8G \left(\frac{\pi}{2} r_2^4\right)} = \frac{\pi r_2^2 \tau_{\max}^2 L}{4G}$$

$$\text{But } \pi r_2^2 = A \quad \therefore U_2 = \frac{A \tau_{\max}^2 L}{4G}$$

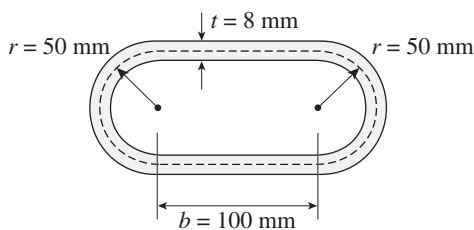
RATIO

$$\frac{U_1}{U_2} = 2 \quad \leftarrow$$

**Problem 3.10-6** Calculate the shear stress  $\tau$  and the angle of twist  $\phi$  (in degrees) for a steel tube ( $G = 76 \text{ GPa}$ ) having the cross section shown in the figure. The tube has length  $L = 1.5 \text{ m}$  and is subjected to a torque  $T = 10 \text{ kN} \cdot \text{m}$ .



**Solution 3.10-6** Steel tube



$$G = 76 \text{ GPa}$$

$$L = 1.5 \text{ m}$$

$$T = 10 \text{ kN} \cdot \text{m}$$

$$A_m = \pi r^2 + 2br$$

$$A_m = \pi(50 \text{ mm})^2 + 2(100 \text{ mm})(50 \text{ mm})$$

$$= 17,850 \text{ mm}^2$$

$$L_m = 2b + 2\pi r$$

$$= 2(100 \text{ mm}) + 2\pi(50 \text{ mm})$$

$$= 514.2 \text{ mm}$$

$$J = \frac{4tA_m^2}{L_m} = \frac{4(8 \text{ mm})(17,850 \text{ mm}^2)^2}{514.2 \text{ mm}}$$

$$= 19.83 \times 10^6 \text{ mm}^4$$

SHEAR STRESS

$$\tau = \frac{T}{2tA_m} = \frac{10 \text{ kN} \cdot \text{m}}{2(8 \text{ mm})(17,850 \text{ mm}^2)}$$

$$= 35.0 \text{ MPa} \quad \leftarrow$$

ANGLE OF TWIST

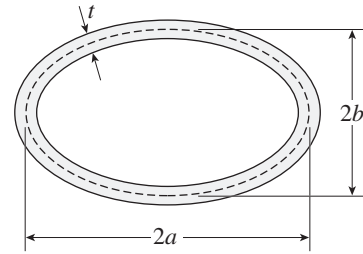
$$\phi = \frac{TL}{GJ} = \frac{(10 \text{ kN} \cdot \text{m})(1.5 \text{ m})}{(76 \text{ GPa})(19.83 \times 10^6 \text{ mm}^4)}$$

$$= 0.00995 \text{ rad}$$

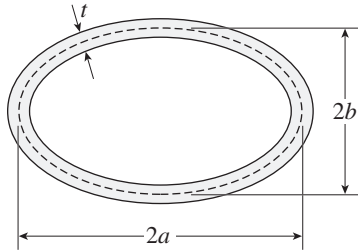
$$= 0.570^\circ \quad \leftarrow$$

**Problem 3.10-7** A thin-walled steel tube having an elliptical cross section with constant thickness  $t$  (see figure) is subjected to a torque  $T = 18$  k-in.

Determine the shear stress  $\tau$  and the rate of twist  $\theta$  (in degrees per inch) if  $G = 12 \times 10^6$  psi,  $t = 0.2$  in.,  $a = 3$  in., and  $b = 2$  in. (Note: See Appendix D, Case 16, for the properties of an ellipse.)



**Solution 3.10-7 Elliptical tube**



$T = 18$  k-in.

$G = 12 \times 10^6$  psi

$t = \text{constant}$

$t = 0.2$  in.  $a = 3.0$  in.  $b = 2.0$  in.

FROM APPENDIX D, CASE 16:

$$A_m = \pi ab = \pi(3.0 \text{ in.})(2.0 \text{ in.}) = 18.850 \text{ in.}^2$$

$$L_m \approx \pi [1.5(a + b) - \sqrt{ab}]$$

$$= \pi [1.5(5.0 \text{ in.}) - \sqrt{6.0 \text{ in.}^2}] = 15.867 \text{ in.}$$

$$J = \frac{4tA_m^2}{L_m} = \frac{4(0.2 \text{ in.})(18.850 \text{ in.}^2)^2}{15.867 \text{ in.}}$$

$$= 17.92 \text{ in.}^4$$

SHEAR STRESS

$$\tau = \frac{T}{2tA_m} = \frac{18 \text{ k-in.}}{2(0.2 \text{ in.})(18.850 \text{ in.}^2)}$$

$$= 2390 \text{ psi} \quad \leftarrow$$

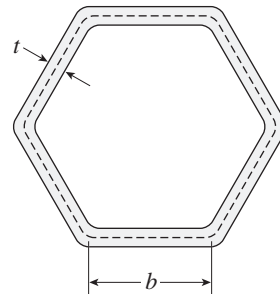
ANGLE OF TWIST PER UNIT LENGTH (RATE OF TWIST)

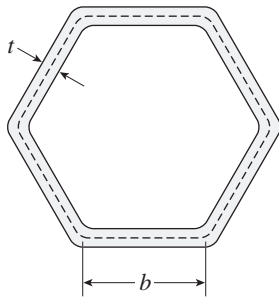
$$\theta = \frac{\phi}{L} = \frac{T}{GJ} = \frac{18 \text{ k-in.}}{(12 \times 10^6 \text{ psi})(17.92 \text{ in.}^4)}$$

$$\theta = 83.73 \times 10^{-6} \text{ rad/in.} = 0.0048^\circ/\text{in.} \quad \leftarrow$$

**Problem 3.10-8** A torque  $T$  is applied to a thin-walled tube having a cross section in the shape of a regular hexagon with constant wall thickness  $t$  and side length  $b$  (see figure).

Obtain formulas for the shear stress  $\tau$  and the rate of twist  $\theta$ .



**Solution 3.10-8 Regular hexagon**

$b$  = Length of side

$t$  = Thickness

$$L_m = 6b$$

FROM APPENDIX D, CASE 25:

$$\beta = 60^\circ \quad n = 6$$

$$\begin{aligned} A_m &= \frac{nb^2}{4} \cot \frac{\beta}{2} = \frac{6b^2}{4} \cot 30^\circ \\ &= \frac{3\sqrt{3}b^2}{2} \end{aligned}$$

SHEAR STRESS

$$\tau = \frac{T}{2tA_m} = \frac{T\sqrt{3}}{9b^2t} \quad \leftarrow$$

ANGLE OF TWIST PER UNIT LENGTH (RATE OF TWIST)

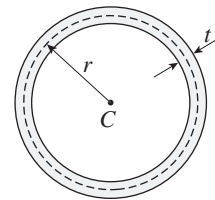
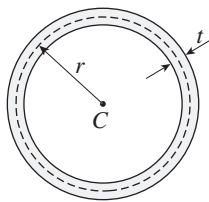
$$J = \frac{4A_m^2 t}{\int_0^{L_m} \frac{d_s}{t}} = \frac{4A_m^2 t}{L_m} = \frac{9b^3 t}{2}$$

$$\theta = \frac{T}{GJ} = \frac{2T}{G(9b^3 t)} = \frac{2T}{9Gb^3 t} \quad \leftarrow$$

(radians per unit length)

**Problem 3.10-9** Compare the angle of twist  $\phi_1$  for a thin-walled circular tube (see figure) calculated from the approximate theory for thin-walled bars with the angle of twist  $\phi_2$  calculated from the exact theory of torsion for circular bars.

- (a) Express the ratio  $\phi_1/\phi_2$  in terms of the nondimensional ratio  $\beta = r/t$ .  
 (b) Calculate the ratio of angles of twist for  $\beta = 5, 10,$  and  $20$ . What conclusion about the accuracy of the approximate theory do you draw from these results?

**Solution 3.10-9 Thin-walled tube**

APPROXIMATE THEORY

$$\phi_1 = \frac{TL}{GJ} \quad J = 2\pi r^3 t \quad \phi_1 = \frac{TL}{2\pi G r^3 t}$$

EXACT THEORY

$$\phi_2 = \frac{TL}{GI_p} \quad \text{From Eq. (3-17): } I_p = \frac{\pi r t}{2} (4r^2 + t^2)$$

$$\phi_2 = \frac{TL}{GI_p} = \frac{2TL}{\pi G r t (4r^2 + t^2)}$$

(a) RATIO

$$\frac{\phi_1}{\phi_2} = \frac{4r^2 + t^2}{4r^2} = 1 + \frac{t^2}{4r^2}$$

$$\text{Let } \beta = \frac{r}{t} \quad \frac{\phi_1}{\phi_2} = 1 + \frac{1}{4\beta^2} \quad \leftarrow$$

(b)

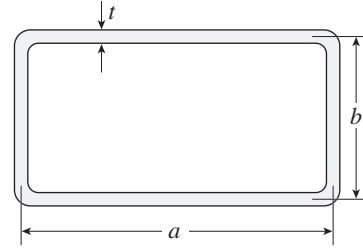
$\beta$	$\phi_1/\phi_2$
5	1.0100
10	1.0025
20	1.0006

As the tube becomes thinner and  $\beta$  becomes larger, the ratio  $\phi_1/\phi_2$  approaches unity. Thus, the thinner the tube, the more accurate the approximate theory becomes.

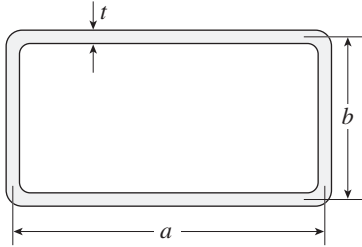
**Problem 3.10-10** A thin-walled rectangular tube has uniform thickness  $t$  and dimensions  $a \times b$  to the median line of the cross section (see figure).

How does the shear stress in the tube vary with the ratio  $\beta = a/b$  if the total length  $L_m$  of the median line of the cross section and the torque  $T$  remain constant?

From your results, show that the shear stress is smallest when the tube is square ( $\beta = 1$ ).



**Solution 3.10-10 Rectangular tube**



$t$  = thickness (constant)  
 $a, b$  = dimensions of the tube

$$\beta = \frac{a}{b}$$

$$L_m = 2(a + b) = \text{constant}$$

$$T = \text{constant}$$

**SHEAR STRESS**

$$\tau = \frac{T}{2tA_m} \quad A_m = ab = \beta b^2$$

$$L_m = 2b(1 + \beta) = \text{constant}$$

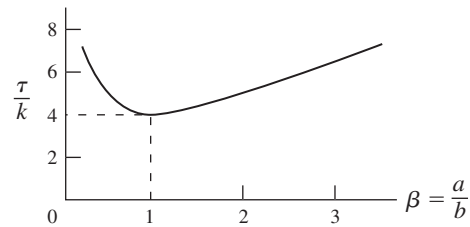
$$b = \frac{L_m}{2(1 + \beta)} \quad A_m = \beta \left[ \frac{L_m}{2(1 + \beta)} \right]^2$$

$$A_m = \frac{\beta L_m^2}{4(1 + \beta)^2}$$

$$\tau = \frac{T}{2tA_m} = \frac{T(4)(1 + \beta)^2}{2t\beta L_m^2} = \frac{2T(1 + \beta)^2}{tL_m^2\beta} \leftarrow$$

$T, t,$  and  $L_m$  are constants.

$$\text{Let } k = \frac{2T}{tL_m^2} = \text{constant} \quad \tau = k \frac{(1 + \beta)^2}{\beta}$$



$$\left(\frac{\tau}{k}\right)_{\min} = 4 \quad \tau_{\min} = \frac{8T}{tL_m^2}$$

From the graph, we see that  $\tau$  is minimum when  $\beta = 1$  and the tube is square.

**ALTERNATE SOLUTION**

$$\tau = \frac{2T}{tL_m^2} \left[ \frac{(1 + \beta)^2}{\beta} \right]$$

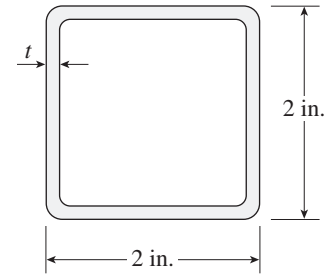
$$\frac{d\tau}{d\beta} = \frac{2T}{tL_m^2} \left[ \frac{\beta(2)(1 + \beta) - (1 + \beta)^2(1)}{\beta^2} \right] = 0$$

$$\text{or } 2\beta(1 + \beta) - (1 + \beta)^2 = 0 \quad \therefore \beta = 1$$

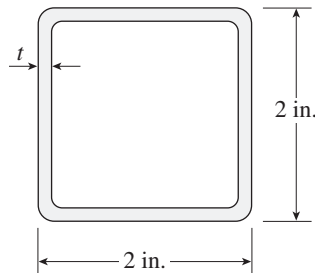
Thus, the tube is square and  $\tau$  is either a minimum or a maximum. From the graph, we see that  $\tau$  is a minimum.

**Problem 3.10-11** A tubular aluminum bar ( $G = 4 \times 10^6$  psi) of square cross section (see figure) with outer dimensions 2 in.  $\times$  2 in. must resist a torque  $T = 3000$  lb-in.

Calculate the minimum required wall thickness  $t_{\min}$  if the allowable shear stress is 4500 psi and the allowable rate of twist is 0.01 rad/ft.



**Solution 3.10-11 Square aluminum tube**



Outer dimensions:

$$2.0 \text{ in.} \times 2.0 \text{ in.}$$

$$G = 4 \times 10^6 \text{ psi}$$

$$T = 3000 \text{ lb-in.}$$

$$\tau_{\text{allow}} = 4500 \text{ psi}$$

$$\theta_{\text{allow}} = 0.01 \text{ rad/ft} = \frac{0.01}{12} \text{ rad/in.}$$

Let  $b$  = outer dimension

$$= 2.0 \text{ in.}$$

Centerline dimension =  $b - t$

$$A_m = (b - t)^2 \quad L_m = 4(b - t)$$

$$J = \frac{4tA_m^2}{L_m} = \frac{4t(b - t)^4}{4(b - t)} = t(b - t)^3$$

THICKNESS  $t$  BASED UPON SHEAR STRESS

$$\tau = \frac{T}{2tA_m} \quad tA_m = \frac{T}{2\tau} \quad t(b - t)^2 = \frac{T}{2\tau}$$

$$\text{UNITS: } t = \text{in.} \quad b = \text{in.} \quad T = \text{lb-in.} \quad \tau = \text{psi}$$

$$t(2.0 \text{ in.} - t)^2 = \frac{3000 \text{ lb-in.}}{2(4500 \text{ psi})} = \frac{1}{3} \text{ in.}^3$$

$$3t(2 - t)^2 - 1 = 0$$

$$\text{Solve for } t: t = 0.0915 \text{ in.}$$

THICKNESS  $t$  BASED UPON RATE OF TWIST

$$\theta = \frac{T}{GJ} = \frac{T}{Gt(b - t)^3} \quad t(b - t)^3 = \frac{T}{G\theta}$$

$$\text{UNITS: } t = \text{in.} \quad G = \text{psi} \quad \theta = \text{rad/in.}$$

$$t(2.0 \text{ in.} - t)^3 = \frac{3000 \text{ lb-in.}}{(4 \times 10^6 \text{ psi})(0.01/12 \text{ rad/in.})}$$

$$= \frac{9}{10}$$

$$10t(2 - t)^3 - 9 = 0$$

Solve for  $t$ :

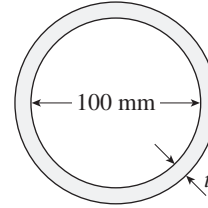
$$t = 0.140 \text{ in.}$$

ANGLE OF TWIST GOVERNS

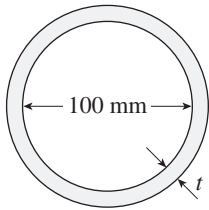
$$t_{\min} = 0.140 \text{ in.} \quad \leftarrow$$

**Problem 3.10-12** A thin tubular shaft of circular cross section (see figure) with inside diameter 100 mm is subjected to a torque of  $5000 \text{ N} \cdot \text{m}$ .

If the allowable shear stress is 42 MPa, determine the required wall thickness  $t$  by using (a) the approximate theory for a thin-walled tube, and (b) the exact torsion theory for a circular bar.



**Solution 3.10-12 Thin tube**



$$T = 5,000 \text{ N} \cdot \text{m} \quad d_1 = \text{inner diameter} = 100 \text{ mm}$$

$$\tau_{\text{allow}} = 42 \text{ MPa}$$

$t$  is in millimeters.

$$r = \text{Average radius}$$

$$= 50 \text{ mm} + \frac{t}{2}$$

$$r_1 = \text{Inner radius}$$

$$= 50 \text{ mm}$$

$$r_2 = \text{Outer radius}$$

$$= 50 \text{ mm} + t \quad A_m = \pi r^2$$

(a) APPROXIMATE THEORY

$$\tau = \frac{T}{2tA_m} = \frac{T}{2t(\pi r^2)} = \frac{T}{2\pi r^2 t}$$

$$42 \text{ MPa} = \frac{5,000 \text{ N} \cdot \text{m}}{2\pi \left(50 + \frac{t}{2}\right)^2 t}$$

or

$$t \left(50 + \frac{t}{2}\right)^2 = \frac{5,000 \text{ N} \cdot \text{m}}{2\pi(42 \text{ MPa})} = \frac{5 \times 10^6}{84\pi} \text{ mm}^3$$

Solve for  $t$ :

$$t = 6.66 \text{ mm} \quad \leftarrow$$

(b) EXACT THEORY

$$\tau = \frac{Tr_2}{I_p} \quad I_p = \frac{\pi}{2}(r_2^4 - r_1^4) = \frac{\pi}{2}[(50 + t)^4 - (50)^4]$$

$$42 \text{ MPa} = \frac{(5,000 \text{ N} \cdot \text{m})(50 + t)}{\frac{\pi}{2}[(50 + t)^4 - (50)^4]}$$

$$\frac{(50 + t)^4 - (50)^4}{50 + t} = \frac{(5,000 \text{ N} \cdot \text{m})(2)}{(\pi)(42 \text{ MPa})}$$

$$= \frac{5 \times 10^6}{21\pi} \text{ mm}^3$$

Solve for  $t$ :

$$t = 7.02 \text{ mm} \quad \leftarrow$$

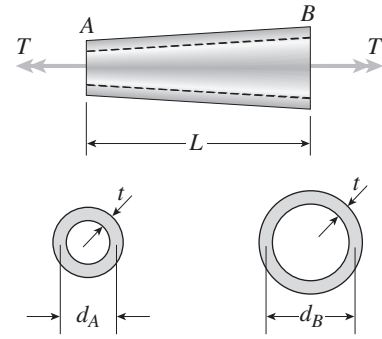
The approximate result is 5% less than the exact result. Thus, the approximate theory is nonconservative and should only be used for thin-walled tubes.

**Problem 3.10-13** A long, thin-walled tapered tube  $AB$  of circular cross section (see figure) is subjected to a torque  $T$ . The tube has length  $L$  and constant wall thickness  $t$ . The diameter to the median lines of the cross sections at the ends  $A$  and  $B$  are  $d_A$  and  $d_B$ , respectively.

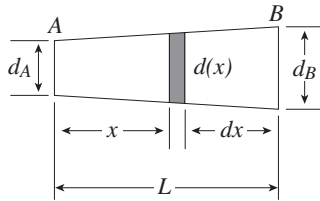
Derive the following formula for the angle of twist of the tube:

$$\phi = \frac{2TL}{\pi Gt} \left( \frac{d_A + d_B}{d_A^2 d_B^2} \right)$$

*Hint:* If the angle of taper is small, we may obtain approximate results by applying the formulas for a thin-walled prismatic tube to a differential element of the tapered tube and then integrating along the axis of the tube.



### Solution 3.10-13 Thin-walled tapered tube



$t$  = thickness

$d_A$  = average diameter at end  $A$

$d_B$  = average diameter at end  $B$

$T$  = torque

$d(x)$  = average diameter at distance  $x$  from end  $A$ .

$$d(x) = d_A + \left( \frac{d_B - d_A}{L} \right) x$$

$$J = 2\pi r^3 t = \frac{\pi d^3 t}{4}$$

$$J(x) = \frac{\pi t}{4} [d(x)]^3 = \frac{\pi t}{4} \left[ d_A + \left( \frac{d_B - d_A}{L} \right) x \right]^3$$

For element of length  $dx$ :

$$d\phi = \frac{T dx}{GJ(x)} = \frac{4T dx}{G\pi t \left[ d_A + \left( \frac{d_B - d_A}{L} \right) x \right]^3}$$

For entire tube:

$$\phi = \frac{4T}{\pi Gt} \int_0^L \frac{dx}{\left[ d_A + \left( \frac{d_B - d_A}{L} \right) x \right]^3}$$

From table of integrals (see Appendix C):

$$\int \frac{dx}{(a + bx)^3} = -\frac{1}{2b(a + bx)^2}$$

$$\phi = \frac{4T}{\pi Gt} \left[ -\frac{1}{2 \left( \frac{d_B - d_A}{L} \right) \left( d_A + \frac{d_B - d_A}{L} \cdot x \right)^2} \right]_0^L$$

$$= \frac{4T}{\pi Gt} \left[ -\frac{L}{2(d_B - d_A)d_B^2} + \frac{L}{2(d_B - d_A)d_A^2} \right]$$

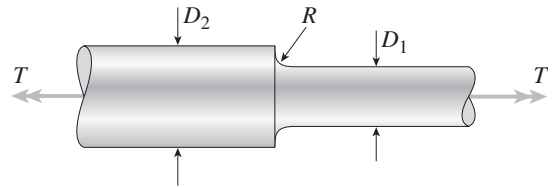
$$\phi = \frac{2TL}{\pi Gt} \left( \frac{d_A + d_B}{d_A^2 d_B^2} \right) \leftarrow$$

### Stress Concentrations in Torsion

The problems for Section 3.11 are to be solved by considering the stress-concentration factors.

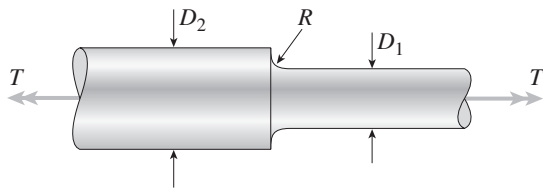
**Problem 3.11-1** A stepped shaft consisting of solid circular segments having diameters  $D_1 = 2.0$  in. and  $D_2 = 2.4$  in. (see figure) is subjected to torques  $T$ . The radius of the fillet is  $R = 0.1$  in.

If the allowable shear stress at the stress concentration is 6000 psi, what is the maximum permissible torque  $T_{\max}$ ?



Probs. 3.11-1 through 3.11-5

#### Solution 3.11-1 Stepped shaft in torsion



$$D_1 = 2.0 \text{ in.}$$

$$D_2 = 2.4 \text{ in.}$$

$$R = 0.1 \text{ in.}$$

$$\tau_{\text{allow}} = 6000 \text{ psi}$$

USE FIG. 3-48 FOR THE STRESS-CONCENTRATION FACTOR

$$\frac{R}{D_1} = \frac{0.1 \text{ in.}}{2.0 \text{ in.}} = 0.05 \quad \frac{D_2}{D_1} = \frac{2.4 \text{ in.}}{2.0 \text{ in.}} = 1.2$$

$$K \approx 1.52 \quad \tau_{\max} = K\tau_{\text{nom}} = K\left(\frac{16 T_{\max}}{\pi D_1^3}\right)$$

$$T_{\max} = \frac{\pi D_1^3 \tau_{\max}}{16K}$$

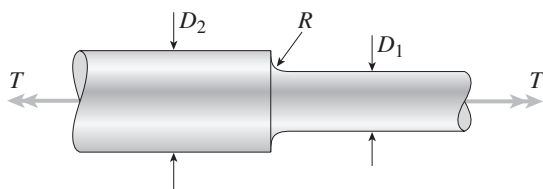
$$= \frac{\pi(2.0 \text{ in.})^3(6000 \text{ psi})}{16(1.52)} = 6200 \text{ lb-in.}$$

$$\therefore T_{\max} \approx 6200 \text{ lb-in.} \quad \leftarrow$$

**Problem 3.11-2** A stepped shaft with diameters  $D_1 = 40$  mm and  $D_2 = 60$  mm is loaded by torques  $T = 1100$  N · m (see figure).

If the allowable shear stress at the stress concentration is 120 MPa, what is the smallest radius  $R_{\min}$  that may be used for the fillet?

#### Solution 3.11-2 Stepped shaft in torsion



$$D_1 = 40 \text{ mm}$$

$$D_2 = 60 \text{ mm}$$

$$T = 1100 \text{ N} \cdot \text{m}$$

$$\tau_{\text{allow}} = 120 \text{ MPa}$$

USE FIG. 3-48 FOR THE STRESS-CONCENTRATION FACTOR

$$\tau_{\max} = K\tau_{\text{nom}} = K\left(\frac{16T}{\pi D_1^3}\right)$$

$$K = \frac{\pi D_1^3 \tau_{\max}}{16T} = \frac{\pi(40 \text{ mm})^3(120 \text{ MPa})}{16(1100 \text{ N} \cdot \text{m})} = 1.37$$

$$\frac{D_2}{D_1} = \frac{60 \text{ mm}}{40 \text{ mm}} = 1.5$$

From Fig. (3-48) with  $\frac{D_2}{D_1} = 1.5$  and  $K = 1.37$ ,

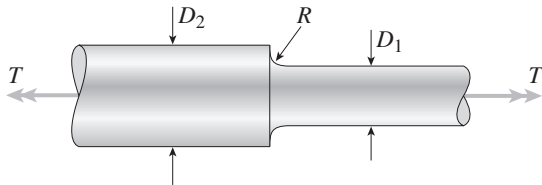
we get  $\frac{R}{D_1} \approx 0.10$

$$\therefore R_{\min} \approx 0.10(40 \text{ mm}) = 4.0 \text{ mm} \quad \leftarrow$$

**Problem 3.11-3** A full quarter-circular fillet is used at the shoulder of a stepped shaft having diameter  $D_2 = 1.0$  in. (see figure). A torque  $T = 500$  lb-in. acts on the shaft.

Determine the shear stress  $\tau_{\max}$  at the stress concentration for values as follows:  $D_1 = 0.7, 0.8,$  and  $0.9$  in. Plot a graph showing  $\tau_{\max}$  versus  $D_1$ .

**Solution 3.11-3 Stepped shaft in torsion**



$D_1$ (in.)	$D_2/D_1$	$R$ (in.)	$R/D_1$	$K$	$\tau_{\max}$ (psi)
0.7	1.43	0.15	0.214	1.20	8900
0.8	1.25	0.10	0.125	1.29	6400
0.9	1.11	0.05	0.056	1.41	4900

$$D_2 = 1.0 \text{ in.}$$

$$T = 500 \text{ lb-in.}$$

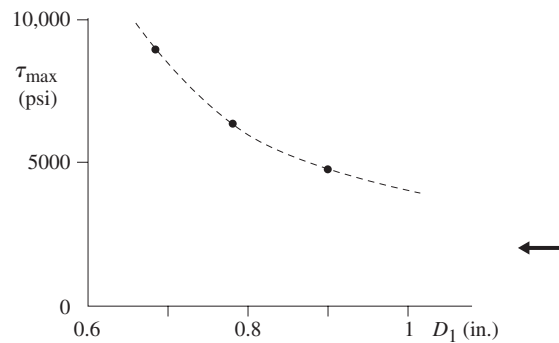
$$D_1 = 0.7, 0.8, \text{ and } 0.9 \text{ in.}$$

Full quarter-circular fillet ( $D_2 = D_1 + 2R$ )

$$R = \frac{D_2 - D_1}{2} = 0.5 \text{ in.} - \frac{D_1}{2}$$

USE FIG. 3-48 FOR THE STRESS-CONCENTRATION FACTOR

$$\begin{aligned} \tau_{\max} &= K\tau_{\text{nom}} = K\left(\frac{16T}{\pi D_1^3}\right) \\ &= K\frac{16(500 \text{ lb-in.})}{\pi D_1^3} = 2546\frac{K}{D_1^3} \end{aligned}$$

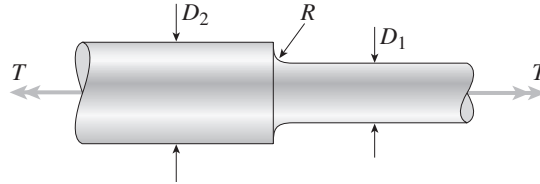


Note that  $\tau_{\max}$  gets smaller as  $D_1$  gets larger, even though  $K$  is increasing.

**Problem 3.11-4** The stepped shaft shown in the figure is required to transmit 600 kW of power at 400 rpm. The shaft has a full quarter-circular fillet, and the smaller diameter  $D_1 = 100$  mm.

If the allowable shear stress at the stress concentration is 100 MPa, at what diameter  $D_2$  will this stress be reached? Is this diameter an upper or a lower limit on the value of  $D_2$ ?

**Solution 3.11-4 Stepped shaft in torsion**



$$P = 600 \text{ kW}$$

$$n = 400 \text{ rpm}$$

Full quarter-circular fillet

$$\text{POWER } P = \frac{2\pi nT}{60} \quad (\text{Eq. 3-42 of Section 3.7})$$

$$P = \text{watts} \quad n = \text{rpm} \quad T = \text{Newton meters}$$

$$T = \frac{60P}{2\pi n} = \frac{60(600 \times 10^3 \text{ W})}{2\pi(400 \text{ rpm})} = 14,320 \text{ N} \cdot \text{m}$$

USE FIG. 3-48 FOR THE STRESS-CONCENTRATION FACTOR

$$\tau_{\max} = K\tau_{\text{nom}} = K\left(\frac{16T}{\pi D_1^3}\right)$$

$$K = \frac{\tau_{\max}(\pi D_1^3)}{16T}$$

$$= \frac{(100 \text{ MPa})(\pi)(100 \text{ mm})^3}{16(14,320 \text{ N} \cdot \text{m})} = 1.37$$

Use the dashed line for a full quarter-circular fillet.

$$\frac{R}{D_1} \approx 0.075 \quad R \approx 0.075 D_1 = 0.075 (100 \text{ mm}) \\ = 7.5 \text{ mm}$$

$$D_2 = D_1 + 2R = 100 \text{ mm} + 2(7.5 \text{ mm}) = 115 \text{ mm}$$

$$\therefore D_2 \approx 115 \text{ mm} \quad \leftarrow$$

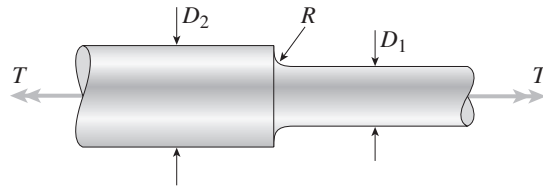
This value of  $D_2$  is a *lower limit*  $\leftarrow$

(If  $D_2$  is less than 115 mm,  $R/D_1$  is smaller,  $K$  is larger, and  $\tau_{\max}$  is larger, which means that the allowable stress is exceeded.)

**Problem 3.11-5** A stepped shaft (see figure) has diameter  $D_2 = 1.5$  in. and a full quarter-circular fillet. The allowable shear stress is 15,000 psi and the load  $T = 4800$  lb-in.

What is the smallest permissible diameter  $D_1$ ?

**Solution 3.11-5 Stepped shaft in torsion**



$$D_2 = 1.5 \text{ in.}$$

$$\tau_{\text{allow}} = 15,000 \text{ psi}$$

$$T = 4800 \text{ lb-in.}$$

Full quarter-circular fillet  $D_2 = D_1 + 2R$

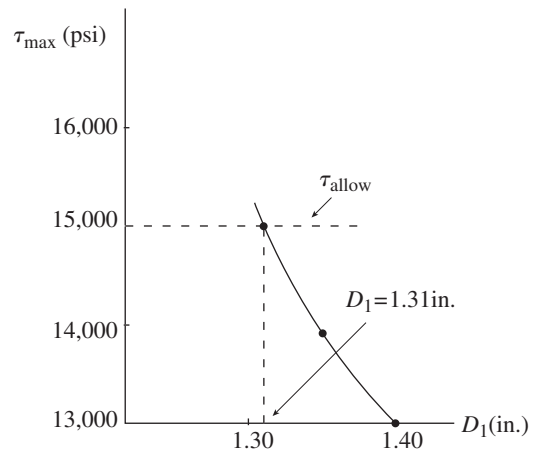
$$R = \frac{D_2 - D_1}{2} = 0.75 \text{ in.} - \frac{D_1}{2}$$

USE FIG. 3-48 FOR THE STRESS-CONCENTRATION FACTOR

$$\begin{aligned} \tau_{\text{max}} &= K\tau_{\text{nom}} = K\left(\frac{16T}{\pi D_1^3}\right) \\ &= \frac{K}{D_1^3} \left[ \frac{16(4800 \text{ lb-in.})}{\pi} \right] \\ &= 24,450 \frac{K}{D_1^3} \end{aligned}$$

Use trial-and-error. Select trial values of  $D_1$

$D_1$ (in.)	$R$ (in.)	$R/D_1$	$K$	$\tau_{\text{max}}$ (psi)
1.30	0.100	0.077	1.38	15,400
1.35	0.075	0.056	1.41	14,000
1.40	0.050	0.036	1.46	13,000



From the graph, minimum  $D_1 \approx 1.31$  in. ←

